On some aspects of the McKay correspondence

Luca Scala

"Les objets concernés par cet article
sont essentiellement les solides platoniciens"
G. Gonzalez-Sprinberg and J. L. Verdier

1 Introduction

When we quotient $\mathbb{C}^2$ by a finite subgroup G of $SL(2,\mathbb{C})$, and we take a minimal resolution Y of the kleinian singularity $\mathbb{C}^2/G$, then Y is a crepant resolution and the exceptional locus consists of a bunch of curves, whose dual graph is a Dynkin diagram of the kind A_n, D_n, E_6, E_7, E_8. In the eighties, McKay noticed that the Dynkin diagrams arising from resolutions of kleinian singularities are in tight connection with the representations of G. In the first part we will explain the McKay correspondence and its key generalization by means of K-theory, due to Gonzalez-Sprinberg and Verdier. The latter point of view opens the way to the modern derived McKay correspondence, due to Bridgeland-King-Reid. We will then see some applications of the BKR theorem to the geometry of Hilbert schemes of points, due to Haiman, and some other consequences related to the cohomology of tautological bundles. In all the exposition, we will always work with algebraic varieties over $\mathbb{C}$; moreover, we will always suppose that all singular varieties are normal.

2 Rational double points

The objects of interest in this section are certain classes of singularities of surfaces and their resolutions. If X is an algebraic variety, we denote with X_{reg} and with X_{sing} the open set of regular points and the closed set of singular points, respectively. We recall the definition of resolution of singularities.

Definition 2.1. Let X an algebraic variety. A resolution of singularities of X is a smooth variety Y, equipped with a proper birational morphism $\mu : Y \longrightarrow X$, such that μ induces an isomorphism between $Y \setminus \mu^{-1}(X_{\text{sing}})$ and X_{reg}. The set $Exc(\mu) := \mu^{-1}(X_{\text{sing}})$, where μ fails to be an isomorphism, is called the exceptional locus.

We say that an algebraic variety X has *rational singularities* if there is a resolution $\mu : Y \longrightarrow X$ such that $\mathbf{R}\mu_*\mathcal{O}_X \simeq \mathcal{O}_Y$, or equivalently, such that $\mu_*\mathcal{O}_Y \simeq \mathcal{O}_X$, and the higher direct images of the structural sheaf of Y vanish: $R^i\mu_*\mathcal{O}_X \simeq 0$. We are interested in (germs of) isolated singularities of surfaces, that is, we consider a small neighbourhood of a surface X around the only singular point x. We denote the germ with (X,x). For such a singularity consider the maximal ideal $\mathfrak{m}_x$ in the local ring $\mathcal{O}_{X,x}$: the Zariski cotangent space of X at x is then $\mathfrak{m}_x/\mathfrak{m}_x^2$. For an isolated singularity (X,x), being rational can be rephrased in terms of complex analytic geometry as follows:

Definition 2.2. [27], [6]. A germ of n-dimensional isolated singularity (X,x) is rational if and only if for all regular holomorphic n-form $\sigma \in H^0(X \setminus \{0\}, \Omega^n_X)$ on $X \setminus \{0\}$, the pull-back $\mu^*\sigma \in H^0(Y \setminus E, \Omega^n_Y)$

extends to a regular holomorphic form on the whole Y. This is equivalent to saying that any holomorphic n-form σ defined in a deleted neighbourhood $U \setminus \{x\}$ of x is square integrable around x, that is

$$\int_{U'} \sigma \wedge \bar{\sigma} < +\infty$$

for U' sufficiently small relatively compact.

Germs of isolated rational surface singularities have been extensively studied by Artin in [1]: among other results, he proves such an isolated rational surface singularity has multiplicity exactly $\dim \mathfrak{m}_x/\mathfrak{m}_x^2 - 1$. Since one can always embed any germ (X, x) in its tangent space at the point x, one has consequently that a rational double point is always embeddable in $\mathbb{C}^3$. Hence the isolated surface singularities that we are interested in are all of the form $(X, x) = (V(f), 0)$, where $f \in \mathbb{C}[x, y, z]$ is a polynomial in 3 variables, with $\nabla f(0) = 0$.

Remark 2.3. In this case, if Y is a resolution of singularities, then the exceptional set with the reduced structure $E = \mu^{-1}(0)_{\mathrm{red}}$ is a divisor (necessarily a curve in Y) and it is always connected (by Zariski main theorem, since X is normal). However E can be reducible. We will write $E = \cup_i C_i$, where C_i are the irreducible components.

Definition 2.4. Let X an n-dimensional algebraic variety. Consider the open immersion $j : X_{\mathrm{reg}} \hookrightarrow X$. Consider the sheaf $\omega_X := j_* \Omega^n_{X_{\mathrm{reg}}}$ and suppose that it is a line bundle. A resolution of singularities $\mu : Y \longrightarrow X$ is called *crepant*[1] if $\mu^* \omega_X \simeq \omega_Y$, where $\omega_Y := \Omega^n_Y$ is the canonical line bundle of Y. For an isolated singularity (X, x) this means that for any holomorphic n-form σ defined on a heighbourhood of x, the form $\mu^*\sigma$ is a holomorphic n-form on $\mu^{-1}(U) \setminus E$ which can be extended to a holomorphic n-form to $\mu^{-1}(U)$ without zero on $\mu^{-1}(U)$.

Remark 2.5. We will say that the resolution $\mu : Y \longrightarrow X$ is *minimal* if it does not factorize through another resolution $\mu' : Y' \longrightarrow X$. It follows that if the germ of surface singularity (X, x) is rational and the resolution $\mu : Y \longrightarrow X$ is minimal, then it is crepant.

Example 2.6. Consider the polynomial $f = x^2 + y^2 - z^2$ (figure 1). The surface $V(f)$ has a singularity at the origin. The singularity is rational, as it can be seen like follows. Since $df = 2xdx + 2ydy - 2zdz = 0$ on X, the differential form on X: $\sigma = -dx \wedge dy/2z = -dy \wedge dz/2y = dx \wedge dz/2y$ is a well defined rational form on X; moreover it is regular and nondegenerate at every nonsingular point of X, hence it is a volume form on $X \setminus \{0\}$. A resolution of X can be obtained considering the blow-up $h : \mathrm{Bl}_0\mathbb{C}^3 \longrightarrow \mathbb{C}^3$ of the origin in $\mathbb{C}^3$, and taking Y as the strict transform of X, that is, the Zariski closure of $h^{-1}(X \setminus \{x\})$. On can easily prove that the form $h^*\sigma$, defined on $Y \setminus E$, can be extended to the whole Y as a volume form. The blow-up has indeed 3-charts; one of them (the other are analogous) has coordinates λ, μ, z, with $x = \lambda z$, $y = \mu z$, and Y is defined on this chart by $\lambda^2 + \mu^2 - 1 = 0$. The exceptional divisor is the circle $E = Y \cap \{z = 0\}$. The differential form $\tau = -d\mu \wedge dz/2\lambda = d\lambda \wedge dz/2\mu$ is a rational volume form on Y, coinciding with $h^*\sigma$ on $Y \setminus E$.

Example 2.7. Consider the polynomial $f = x^2 - y^2 z - z^3$ (figure 2). The surface $V(f)$ has an isolated rational singularity at the origin. In order to solve it, we need two blow-ups. After the first one, there will be three distinct singular points in the exceptional divisor. Blowing-up the three of them at once, we get the resolution Y. The exceptional divisor E is a union of four rational curves C_i.

Definition 2.8. If (X, x) is germ of an isolated rational surface singlarity and $\mu : Y \longrightarrow X$ a minimal resolution with exceptional divisor $E = \sum_i C_i$, the cycle $W = \sum_i r_i C_i$ given by the nonreduced scheme $W := \mu^{-1}(x)$ is called the *fundamental cycle*.

[1]This means that there is no *discrepancy* between ω_Y and $\mu^*\omega_X$

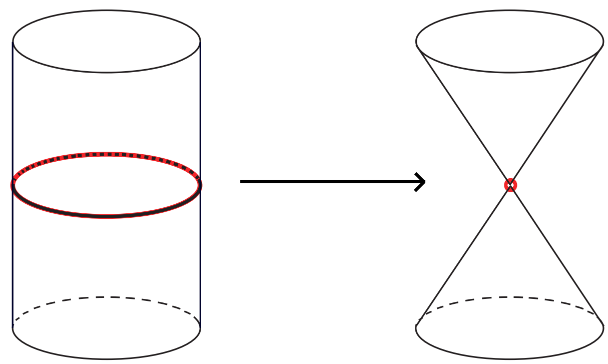


Figure 1: Minimal resolution of the A_1-singularity $x^2 + y^2 - z^2 = 0$.

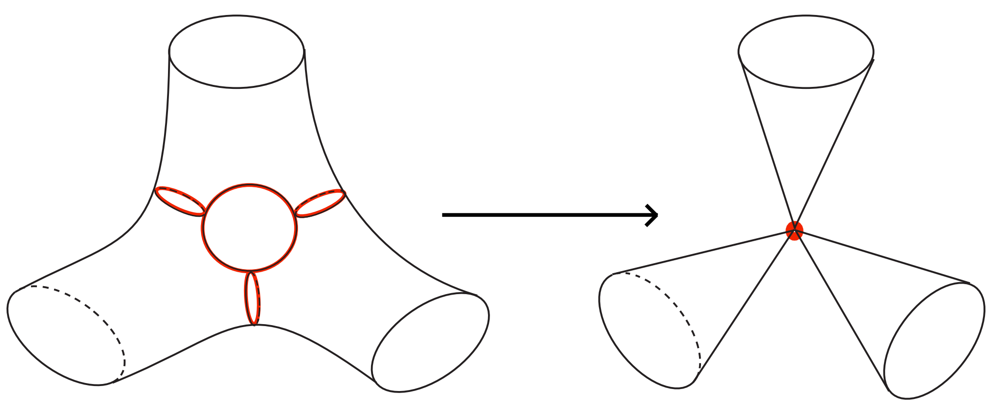
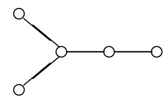
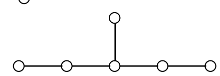
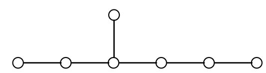
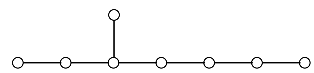


Figure 2: Minimal resolution of the D_4-singularity $x^2 - y^2z - z^3 = 0$.

Remark 2.9. If (X, x) is a germ of rational surface singularities and $\mu : Y \longrightarrow X$ is a minimal resolution, we can understand completely the kind of curves C_i appearing as irreducible components of E and the structure of their intersections [2, chapter 3, §2,3]. Indeed

- The autointersection C_i^2 of each curve is -2. This is equivalent to the fact that all curve are rational, and actually isomophic to $\mathbb{P}_1$.
- If we draw a point for each curve C_i and a line between points if the two corresponding curve intersect, the diagram we obtain are all and only the following Dynkin diagrams. It is clear that isomoprhic germ singularities will generate the same diagrams, so the following is a classification

of isomophism classes of rational double points. The matrix $(C_i \cdot C_j)_{ij}$, whose information is equivalent to the information given by the diagrams, is called the *intersection matrix*.

A_n	$x^2 + y^2 + z^{n+1}$	
D_n	$x^2 + y^2 z + z^{n-1}$	
E_6	$x^2 + y^3 + z^4$	
E_7	$x^2 + y^3 + yz^3$	
E_8	$x^2 + y^3 + z^5$	

Remark 2.10. Rational double points have many other beautiful and interesting characterizations and connections, not only in terms of algebraic geometry, but also of complex analysis, Lie Groups, differential topology, algebraic topology and fundamental groups, Morse theory, catastrophe theory and many others. See for example [14] and [38].

3 Finite subgroups of $SU(2)$

Interesting isolated surface singularities come from quotients $\mathbb{C}^2/G$, with G a finite group of $SL(2, \mathbb{C})$. Any finite subgroup of $SL(2, \mathbb{C})$ is conjugated to a subgroup of $SU(2)$, the reason being that, by averaging, one can build a G-invariant hermitian metric in $\mathbb{C}^2$. In this section we will say some words on the classification of finite subgroups of $SU(2)$.

As an immediate consequence of its definition, the group $SU(2)$ is diffeomeorphic to the sphere S^3 and hence simply connected. Moreover there is a $2:1$ covering map $\pi : SU(2) \longrightarrow SO(3)$, that realizes it as the universal cover of $SO(3)$, or, in other terms, as $Spin(3)$.

Remark 3.1. Since the only element of order 2 in $SU(2)$ is -1, we have that if G is a finite subgroup fo $SU(2)$, then, up to conjugation, G is a cyclic group of finite order, or $G = \pi^{-1}(G')$ with G' a finite group of $SO(3)$, that is, a *binary polyhedral group*. Indeed if $|G|$ is odd, then $G \cap \ker = \{1\}$ and hence $G \simeq \pi(G)$, and hence it has to be cyclic. Otherwise, if $|G|$ is even, then, by Sylow theorem, it contains a subgroup of order a power of 2, and hence an element of order 2, that is, it has to contain the kernel. Hence, $G = \pi^{-1}\pi(G)$.

Remark 3.2. After the previous remark, to classify, up to conjugation, finite subgroups of $SU(2)$, we just have to classify finite subrgroups of $SO(3)$. Let G a finite subgroup of $SO(3)$. Let p a point of $\mathbb{R}^3$, $p \neq 0$. Then the orbit Gp can be planar or not. If Gp is planar, then G is *cyclic of order* n, or a *dihedral group* (of order $2n$), that is, the symmetry group of a polygon with n sides. On the other hand, if the orbit is not planar, then it is the set of vertices of a regular polyhedron, and G is its symmetry group. Regular polyhedra, or platonic solids have been classified first by Theaetetus (c. 417 B.C. 369 B.C.), a Greek mathematician contemporary to Plato, and the classification has been reported by Plato himself in [34] and by Euclid in the Elements [18] . Since esahedron and octahedron, dodecahedron and icosaheron are dual couples of platonic solids, they have the same symmetry group. Hence, all possible symmetry groups of platonic solids are: the *tetrahedral group*, isomorphic to the alternating group $\mathfrak{A}_4$, with 12 elements, the *octrahedral group*, isomorphic to the symmetric group $\mathfrak{S}_4$, with 24 elements, the *icosahedral group*, isomorphic to $\mathfrak{S}_5$, with 60 elements.

As a consequence of the previous two remarks, we can write down the list of all possible finite subroups G of $SU(2)$ (and of $SL(2,\mathbb{C})$) up to conjugation.

C_n	cyclic	n
BD_{4n}	binary dihedral	$4n$, $n \geq 2$
BT_{24}	binary tetrahedral	24
BO_{48}	binary octahedral	48
BI_{120}	binary icosahedral	120

4 Quotient singularities

Consider now the quotient $\mathbb{C}^2/G$, with G a finite subgroup of $SL(2,\mathbb{C})$. We have that 0 is the unique point with nontrivial stabilizer. Since the action of G on $\mathbb{C}^2 \setminus \{0\}$ is free, the quotient $(\mathbb{C}^2 \setminus \{0\})/G$ is a smooth variety. The topological[2] space $\mathbb{C}^2/G$ can be given the structure of an affine algebraic surface with an isolated singularity in $[0]$.

Remark 4.1. If X is an affine variety with ring of regular functions $A(X)$, then X can be recovered from $A(X)$ taking the spectrum $\operatorname{Spec} A(X)$ of $A(X)$, that is considering all the prime ideals of $A(X)$, if we want the whole scheme structure, and just by taking the maximal spectrum $\operatorname{Max} A(X)$, if we just want to recover the closed points, that is, the structure of algebraic variety.

As a consequence of the previous remark, in order to put on the quotient $\mathbb{C}^2/G$ a structure of affine variety, it is just necessary to assign the algebra of regular functions $A(\mathbb{C}^2/G)$. We remark that the projection $\mathbb{C}^2 \longrightarrow \mathbb{C}^2/G$ has to induce by pull-back a morphism between the algebras of regular functions (as it does for continuous functions)

$$\pi^* : A(\mathbb{C}^2/G) \longrightarrow A(\mathbb{C}^2)^G , \tag{4.1}$$

since we want that the pull-back on $\mathbb{C}^2$ of any regular function on $\mathbb{C}^2/G$ has to be automatically G-invariant. We require that the pull-back (4.1) is actually an isomorphism. Hence, as an algebraic scheme,

$$\mathbb{C}^2/G := \operatorname{Spec} A(\mathbb{C}^2)^G \simeq \operatorname{Spec} \mathbb{C}[x,y]^G .$$

The reassuring thing is that, topologically, the variety underlying $\operatorname{Spec} A(\mathbb{C}^2)^G$ is homeomorphic to the original topological quotient $\mathbb{C}^2/G$. The only thing that remains is the understanding of the invariants $\mathbb{C}[x,y]^G$. This is provided by the following theorem.

Theorem 4.2 (Klein, 1884, [26])**.** *Let G a finite subgroup of $SL(2,\mathbb{C})$. Then the ring of G-invariants $\mathbb{C}[x,y]^G$ is generated by three invariants polynomial $P,Q,R \in \mathbb{C}[x,y]^G$, with a unique relation $S(P,Q,R) = 0$.*

As a consequence of Klein theorem, we have an epimorphism $\mathbb{C}[u,v,w] \twoheadrightarrow \mathbb{C}[x,y]^G$, sending u on P, v on Q and w on R. The kernel is generated by the principal ideal $(S(u,v,w))$. Hence passing to the quotient we get an isomorphism:

$$\mathbb{C}[u,v,w]/(S) \simeq \mathbb{C}[x,y]^G .$$

[2]Here we take the Zariski topology on $\mathbb{C}^2$ and the quotient topology on $\mathbb{C}^2/G$.

As a consequence we get the immersion:

$$\mathbb{C}^2/G \simeq \operatorname{Spec}\mathbb{C}[x,y]^G \simeq \operatorname{Spec}\mathbb{C}[u,v,w]/(S) \hookrightarrow \operatorname{Spec}\mathbb{C}[u,v,w] \simeq \mathbb{C}^3 ,$$

where $\mathbb{C}^2/G$ is embedded in $\mathbb{C}^3$ as the hypersurface of equation $S(u,v,w)=0$. Hence $(\mathbb{C}^2/G,[0]) \simeq (V(S),0)$ is an isolated surface singularity. The quotient singularities of the form $\mathbb{C}^2/G$, with G a finite group of $SL(2,\mathbb{C})$ are called *kleinian singularities.*

Example 4.3. Let $G=\mathbb{Z}_m$, the cyclic group with m elements, acting on $\mathbb{C}^2$ in the following way. If ε is a primitive m-root of unity, then it acts on (x,y) by sending it to $(\varepsilon x, \varepsilon^{-1}y)$. The invariant polynomials $\mathbb{C}[x,y]^G$ are generated by $P=x^m$, $Q=y^m$, $R=xy$, with the relation $R^m=PQ$. Hence $\mathbb{C}^2/G$ can be embedded in $\mathbb{C}^3$ as the hypersurface of equation $w^m=uv$, that, with a change of coordinates, becomes $u^2+v^2+w^m=0$. We remark that it is one of the rational double point listed at page 4, as a A_n-singularity.

Example 4.4. Consider the binary dihedral group BD_{4n}. It is generated by two elements $G=\langle\alpha,\beta\rangle$,

$$\alpha=\begin{pmatrix}\varepsilon & 0\\ 0 & \bar{\varepsilon}\end{pmatrix} \qquad \beta=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

where ε is a primitive $2n$-root of unity; here $\alpha^n=\beta^2=-1$, $\alpha\beta=\beta\alpha^{-1}$. The invariant polynomials are: $P=x^n+y^n$, $Q=x^2y^2$, $R=xy(x^{2n}-y^{2n})$. The relation S is $S(P,Q,R)=R^2-P^2Q+4P^{n+1}$. Hence $\mathbb{C}^2/G$ can be embedded in $\mathbb{C}^3$ as the hypersurface of equation $u^n-v^2w+4w^{n+1}=0$, or, after a change of variable, of equation, $u^2+vw^2+w^{n+1}=0$. This is also listed on page 4 as a D_{n+2} singularity.

It is not a case that the singularities obtained by the previous two examples are rational double points. In general one has:

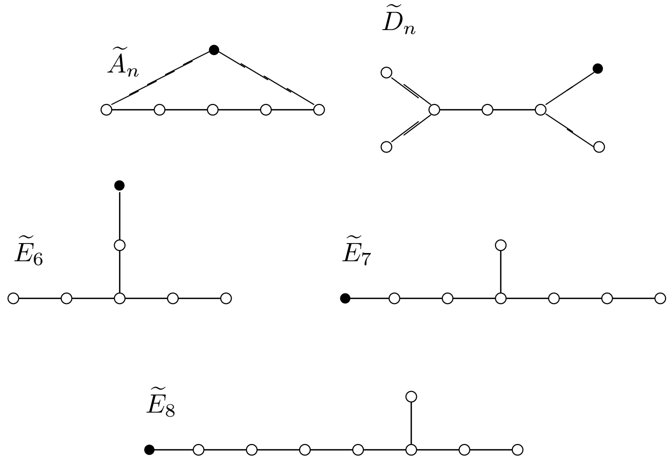
Theorem 4.5 (Du Val, 1934, [11, 12, 13])**.** *If G is a finite subgroup of $SL(2,\mathbb{C})$, the kleinian singularity $\mathbb{C}^2/G$ is a rational double point. For each finite subgroup of G, up to conjugation, we have exactly one isomorphism class of singularities. They correspond to each other in the following way.*

A_n	$x^2+y^2+z^{n+1}$	C_n	*cyclic*
D_n	$x^2+y^2z+z^{n+1}$	$BD_{4(n-2)}$	*binary dihedral*
E_6	$x^2+y^3+z^4$	BT_{24}	*binary tetrahedral*
E_7	$x^2+y^3+yz^3$	BO_{48}	*binary octahedral*
E_8	$x^2+y^3+z^5$	BI_{120}	*binary icosahedral*

See also [37, Chapter IV, §4.3].

5 The McKay correspondence

In the eighties [32, 33] John McKay had the idea to relate, in a purely combinatorial but completely unexpected way, the geometry of a minimal resolution of a kleinian singularity $\mathbb{C}^2/G$, and in particular the intersection graph of the irreducible components of the exceptional divisor, with the irreducible representations of G. In order to be able to explain such a correspondence, we have to introduce the *extended Dynkin diagrams*, obtained by the ADE diagrams, by adding to each of them a point, in the following way.



Remark 5.1. The newly added point is motivated by the following. Consider a minimal resolution $\mu : Y \longrightarrow X = \mathbb{C}^2/G$ of a Kleinian singularity. Let now C_0 the strict transform of a general hyperplane section in X. If $C_1, \ldots, C_n$ are the irreducible components of the exceptional divisor E, whose intersection graph is the old Dynkin diagram, the intersection matrix $\tilde{A}$ of the set of curves $C_0, C_1, \ldots, C_n$ corresponds to the extended Dynkin diagram, with C_0 corresponding to $\bullet$.

Consider now the set $\operatorname{Irr} G = \{\rho_0, \ldots, \rho_n\}$ of irreducible representations of G. Here ρ_0 is the trivial representation. Associate to $\operatorname{Irr} G$ the matrix $A = (a_{ij})$ whose terms a_{ij} are the coefficient of $\mathbb{C}^2 \otimes \rho_j$ in terms of ρ_i:

$$\mathbb{C}^2 \otimes \rho_j = \sum_i a_{ij} \rho_i \ .$$

McKay proved:

Theorem 5.2 (McKay, 1980)**.** *There is a bijection*

$$\operatorname{Irr} G \longleftrightarrow \{C_0, \ldots, C_n\} \tag{5.1}$$

such that $\rho_i \longrightarrow C_i$ *for all* i*, and*

i) $(C_i \cdot C_j) = a_{ij} - 2\delta_{ij}$ *(or* $(C_i \cdot C_j)_{ij} = A - 2\mathrm{id} = \tilde{A}$*);*

ii) $\dim \rho_i = r_i$*, where* $W = \sum_i r_i C_i = \mu^{-1}(0)$ *is the fundamental cycle.*

Remark 5.3. The cohomology classes $[C_i]$ of curves $\{C_0, \ldots, C_n\}$ form a basis of the cohomology $H^*(Y, \mathbb{Z})$.

6 Geometric McKay correspondence

A few years after McKay result, Gonzalez-Sprinberg and Verdier [20] succeeded in giving a geometric construction of McKay correspondence. They actually prove a more general correspondence at the K-theory level, which induces McKay's one. We recall that, for a smooth algebraic variety V, the K-theory $K(V)$ is the ring generated by locally free sheaves (vector bundles) on V with a relation $E = E_1 + E_2$ whenever E is an extension of E_1 and E_2, that is, whenever we have a short exact sequence: $0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0$; the multiplication is given by the tensor product. The use of K-theory allows to to reinterpret the terms of the correspondence. Indeed

- the set of curves $\{C_0, \ldots, C_n\}$ (which actually give information on the second cohomology $H^2(Y, \mathbb{Z})$ of the minimal resolution) is replaced with the larger K-theory ring $K(Y)$. It is not difficult to prove that
$$K(Y) \simeq \mathbb{Z} \oplus \mathrm{Pic}(Y) \simeq \mathbb{Z} \oplus H^2(Y, \mathbb{Z})$$
via the map that associate to a vector bundle E the couple $(\mathrm{rk}\, E, c_1(E))$ given by its rank $\mathrm{rk}\, E \in \mathbb{Z}$ and its first Chern class $c_1(E) \in H^2(Y, \mathbb{Z})$. Hence the ring $K(Y)$ allows to recover the information given by the second cohomology $H^2(Y, \mathbb{Z})$ and its basis $\{C_0, \ldots, C_n\}$.

- Even if $K(\mathbb{C}^2)$ does not provide much information, since it is trivial—$K(\mathbb{C}^2) \simeq \mathbb{Z}$— the G-equivariant K-theory of $\mathbb{C}^2$, that is ring generated by G-equivariant vector bundles (with analogous relations given by extensions), gives the needed informations. Indeed one can prove that the map $K_G(\mathbb{C}^2) \longrightarrow R(G)$, associating to a G-equivariant vector bundle E its 0-fiber $E(0)$, is a ring isomorphism, with inverse $\rho \longrightarrow \mathcal{O}_{\mathbb{C}^2} \otimes_{\mathbb{C}} \rho$.

The geometric construction of the correspondece is now built as follows. Consider the *reduced fibered product* $\mathcal{Z} := (Y \times_X \mathbb{C}^2)_{\mathrm{red}}$. Then one has a (non cartesian) diagram:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "p"] \arrow[d, "q"'] & \mathbb{C}^2 \arrow[d, "\pi"] \\ Y \arrow[r, "\mu"] & X = \mathbb{C}^2/G \end{tikzcd}$$

Remark 6.1. One can prove easily that p and q are birational, while it is more difficult (and it is a key point, as we will see later) to prove that q is flat and finite of degree $|G|$.

Gonzalez-Sprinberg and Verdier define a morphism of groups $\lambda : R(G) \longrightarrow K(Y)$ as a composition:

$$\lambda : R(G) \simeq K_G(\mathbb{C}^2) \xrightarrow{p^*} K_G(\mathcal{Z}) \xrightarrow{q_*^G} K(Y) \ ,$$

that is, for any $\rho \in R(G)$
$$\lambda(\rho) := q_*^G(p^* \mathcal{O}_{\mathbb{C}^2} \otimes_{\mathbb{C}} \rho) = q_*^G(\mathcal{O}_{\mathcal{Z}} \otimes_{\mathbb{C}} \rho) \ ;$$
where q_*^G is the G-invariant push forward, that is, the push-forward followed by the functor of G-fixed points $[-]^G$. The morphism λ is a K-theoretical integral transform of kernel $\mathcal{Z}$. We have the following result, stating the geometric realization of the McKay correspondence.

Theorem 6.2 (Gonzalez-Sprinberg, Verdier, 1983, [20])**.** *The morphism λ is an isomorphism of abelian groups such that:*

i) If $\rho_i \in \mathrm{Irr}\, G$, then $c_1(\lambda(\rho_i)) = [C_i] \in H^2(Y, \mathbb{Z})$;

ii) $c_1(\lambda(\rho_i)) \cdot c_1(\lambda(\rho_j)) = a_{ij}$ for $i \neq j$;

iii) $[W] = \sum_i (\dim \rho_i) c_1(\lambda(\rho_i)) \in H^2(Y, \mathbb{Z})$.

Remark 6.3. Remark that the composition: $\mathrm{Irr}\, G \hookrightarrow R(G) \xrightarrow{\lambda} K(Y) \longrightarrow H^2(Y, \mathbb{Z})$ realizes the classical McKay correspondence (5.1).

Definition 6.4. The sheaves $\mathcal{F}_{\rho_i} := \lambda(\rho_i) = q_*^G(\mathcal{O}_{\mathcal{Z}} \otimes_{\mathbb{C}} \rho_i)$, where ρ_i is an irreducible representation of G, are called *Gonzalez-Sprinberg-Verdier sheaves.*

7 Derived McKay correspondence

The geometric McKay correspondence of Gonzalez-Sprinberg and Verdier is the turning point for more recent developments. Notably, after the work [20], some of the questions that could be raised were how to generalize the result in higher dimensions, to general smooth varieties (instead of $\mathbb{C}^n$) and how to lift it to the derived category level. One of the key difficulties in order to answer these questions is how to replace Y in all generality: the existence of a crepant resolution of singularities is indeed not at all guaranteed in dimension 3 or more [35].

The fundamental point is to consider Y as a *moduli space*, that is a variety parametrizing some kind of objects on $\mathbb{C}^2$. A close look at Gonzalez-Sprinberg construction allows to guess what are the objects that Y could parametrize. In the diagram (6) the reduce fiber product Z inherits a G-action (through the factor $\mathbb{C}^2$); moreover, the morphism $q : Z \longrightarrow Y$ is *flat* and finite of degree $|G|$, as we remarked; finally q is G-invariant. Consequently $Z \subseteq Y \times \mathbb{C}^2$ can be seen as a *flat family over Y of G-equivariant subschemes of $\mathbb{C}^2$ of length $|G|$.*

The precise construction was built by Ito and Nakamura in 1996 [24], [23] for a general smooth quasi-projective variety M, equipped with the action of a finite group G, and goes under the name of Nakamura G-Hilbert scheme $\mathrm{Hilb}^G(M)$.

Definition 7.1. The G-Hilbert scheme $\mathrm{GHilb}(M)$ of G-clusters on M is the scheme representing the functor:

$$\underline{\mathrm{GHilb}(M)} : \mathrm{Sch}/\mathbb{C} \longrightarrow \mathrm{Sets}$$

associating to a scheme S the set

$$\underline{\mathrm{GHilb}(M)}(S) := \{Z \subset S \times M, Z \text{ closed } G\text{-invariant subscheme,}$$
$$\text{flat and finite over } S \text{ such that } H^0(\mathcal{O}_{Z_s}) \simeq \mathbb{C}[G] \text{ for all } s \in S\} \,.$$

The irreducible component of $\mathrm{GHilb}(M)$ containing free G-orbits is called the *Nakamura G-Hilbert scheme*, and it is indicated with $\mathrm{Hilb}^G(M)$.

Remark 7.2. The necessity of taking the irreducible component containing free orbits comes from the fact that, in general, the scheme $\mathrm{GHilb}(M)$ is very bad: it is not irreducible and not even equidimensional. One has a natural morphism $\mu : \mathrm{Hilb}^G(M) \longrightarrow M/G$, called the G-Hilbert-Chow morphism, which sends a free orbit Gx over the class $[x]$; it is birational and dominant.

Remark 7.3. Being a the scheme representing the functor $\underline{\mathrm{GHilb}(M)}$, the scheme $\mathrm{GHilb}(M)$ is a fine moduli space of *G-clusters*, that is, G-invariant subschemes ξ of M of length $|G|$, such that $H^0(\xi)$ is isomorphic to the regular representation $\mathbb{C}[G]$ of G. Hence there is a universal family $\mathcal{Z}$ of G-clusters, $\mathcal{Z} \subseteq \mathrm{GHilb}(M) \times M$. The restriction of $\mathcal{Z}$ to $\mathrm{Hilb}^G(M)$ provides a flat and finite family of G-clusters over $\mathrm{Hilb}^G(M)$.

Denote from now one with Y the Nakamura G-Hilbert scheme $\mathrm{Hilb}^G(M)$. We have the diagram:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "p"] \arrow[d, "q"'] & M \arrow[d, "\pi"] \\ Y \arrow[r, "\mu"] & M/G = X \end{tikzcd}$$

The morphisms p and μ are birational, the morphisms q and π are finite of generic degree $|G|$, q is flat. Remark that p is G-equivariant.

Remark 7.4. An algebraic variety X is said to be Cohen-Macauley if all local rings $\mathcal{O}_{X,x}$, for all points $x \in X$, are Cohen-Macauley. In this case (see [22]) there exists a dualizing sheaf ω_X°, that allows Serre duality. The variety X is said to be Gorenstein (or to have Gorenstein singularities) if it is Cohen-Macauley and the dualizing sheaf is actually a line bundle; in that case $\omega_X^\circ \simeq j_* \omega_{X_{\mathrm{reg}}}$, where j is the open immersion $X_{\mathrm{reg}} \hookrightarrow X$. For the quotient M/G of a smooth variety by a finite group to be Gorenstein, it suffices (and is actually equivalent) that the stabilizer G_x of any points acts on the tangent space $T_x M$ as a subgroup of $SL(T_x M)$. Indeed in this case the canonical line bundle ω_M is preserved by G, and hence it is locally trivial as a G-line bundle; therefore it descends to a line bundle $\omega_{M/G}$ on M/G, which coincide with the canonical line bundle on the smooth points of M/G; it will be isomorphic to the dualizing sheaf $\omega_{M/G}^\circ$ of M/G.

With these premises Bridgeland, King and Reid proved in 2001, under some reasonable hypothesis, a general derived category version of the geometric McKay correspondence.

Theorem 7.5 (Bridgeland-King-Reid, 2001,[5])**.** *Suppose that M is a smooth quasi-projective variety, $G \subseteq \mathrm{Aut}(M)$ is a finite group of automorphism of M and that:*

i) M/G is Gorenstein

ii) $\dim Y \times_{M/G} Y \leq \dim Y + 1$.

Then $Y = \mathrm{Hilb}^G(M)$ is a crepant resolution of M/G and the Fourier-Mukai functor:

$$\mathbf{\Phi} := \mathbf{R}p_* \circ q^* = \mathbf{D}^b(Y) \longrightarrow \mathbf{D}^b_G(M) \tag{7.1}$$

is an equivalence between the bounded derived category of coherent sheaves on Y and the bounded derived category of G-equivariant coherent sheaves on M.

Remark 7.6. The derived equivalence (7.1) was first proved by Kapranov and Vasserot [25] in the classical case of McKay correspondence, $M = \mathbb{C}^2$, $G \subseteq SL(2, \mathbb{C})$. The key point in the proof is in any case the use of Nakamura G-Hilbert scheme.

Remark 7.7. The theorem 7.5 implies that the geometric McKay correspondence holds for three dimensional quotient singularities $\mathbb{C}^3/G$, with $G \subseteq SL(3, \mathbb{C})$. Already in dimension 4, it can be that the hypothesis of 7.5 are not verified. Some quotients $\mathbb{C}^4/G$ do not admit any crepant resolution [35]. In general, the conjectural equivalence $\mathbf{D}^b(Y) \longrightarrow \mathbf{D}_G(\mathbb{C}^n)$ if Y is a crepant resolution of $\mathbb{C}^n/G$, $G \subseteq SL(n, \mathbb{C})$ is called the *derived McKay correspondence conjecture.*

8 Applications and new directions

8.1 Haiman's work

In order to study the $n!$ conjecture, Haiman worked out the situation of the action of the symmetric group on the n-cartesian product of a smooth surface. Let X a smooth quasi-projective surface and consider the cartesian product X^n. The symmetric variety $S^n X$ is the quotient $X^n/\mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group. Consider the Hilbert scheme $X^{[n]}$, parametrizing length n-subschemes of X. The Hilbert-Chow morphism $\mu : X^{[n]} \longrightarrow S^n X$, is defined as $\mu(\xi) = \sum_{x \in X} \mathrm{length}(\mathcal{O}_{\xi,x}) x$. The following facts are well known.

i) The symmetric variety $S^n X$ has rational singularities (see [6], [3]).

ii) By a theorem of Fogarty [19], the Hilbert scheme $X^{[n]}$ is smooth of dimension $2n$ and the Hilbert-Chow morphism provides a crepant resolution of singularities of S^nX.

iii) S^nX is Gorenstein, since the stabilizer of any point x is a subgroup of $SL(T_xX^n)$ and hence the canonical bundle ω_{X^n} is locally trivial as $\mathfrak{S}_n$-sheaf.

iv) μ is a semismall resolution. This follows from works of Briançon [4], Ellingsrud-Stromme [17] or Ellingsrud-Lehn [16] and the stratification of S^nX in terms of partitions of n.

We remark that we are in a situation very similar to the classical McKay correspondence. Moreover almost all the hypothesis of Bridgeland-King-Reid theorem are verified, since S^nX is Gorenstein and the Hilbert-Chow morphism is semismall, and hence $\dim X^{[n]} \times_{S^nX} X^{[n]} \leq \dim X^{[n]} + 1$. It remains to compare the Nakamura $\mathfrak{S}_n$-Hilbert scheme $Y = \mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$ with the Hilbert scheme of points $X^{[n]}$ and to understand what is the universal $\mathfrak{S}_n$-cluster $\mathcal{Z}$. It turns out that the right universal family of $\mathfrak{S}_n$ clusters is provided by Haiman's *isospectral Hilbert scheme*.

Definition 8.1. Let X a smooth quasi-projective algebraic surface. The *isospectral Hilbert scheme* B^n of n points on the surface X is the *reduced* fibered product

$$B^n := (X^{[n]} \times_{S^nX} X^n)_{\mathrm{red}} \ .$$

As a consequence we are give a noncartesian diagram:

$$\begin{tikzcd} B^n \arrow[r, "p"] \arrow[d, "q"'] & X^n \arrow[d, "\pi"] \\ X^{[n]} \arrow[r, "\mu"'] & S^nX \end{tikzcd}$$

with p birational and q finite. Haiman proves the following

Theorem 8.2 (Haiman, 2001, [21]). *1. The isospectral Hilbert scheme is irreducible of dimension $2n$ and can be identified with the blow-up of the union of the pairwise diagonals $\cup_{i<j}\Delta_{ij}$ in X^n: $B^n := \mathrm{Bl}_{\cup_{i<j}\Delta_{ij}} X^n$.*

2. The isospectral Hilbert scheme B^n is normal, Cohen-Macauley and Gorenstein.

Remark 8.3. The Cohen-Macauley property implies the wanted flatness of the morphism q, since a finite surjective morphism between a Cohen-Macauley variety and a smooth one is necessarily flat ([15], chapter 18). Consequently the morphism $q : B^n \longrightarrow X^{[n]}$ is flat and finite of degree $n!$. Moreover, the isospectral Hilbert scheme B^n inherits a $\mathfrak{S}_n$-action, since it is the blow-up of X^n along a closed $\mathfrak{S}_n$-invariant subscheme. This fact implies that B^n is a flat family of $\mathfrak{S}_n$-cluster and gives origin to a map: $\phi : X^{[n]} \longrightarrow \mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$, which allows to compare the two Hilbert schemes. It is now easy to prove that ϕ is an isomorphism, that is, *the Hilbert scheme $X^{[n]}$ can be identified with the Nakamura G-Hilbert scheme* $\mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$ and B^n can be identified with the universal $\mathfrak{S}_n$-cluster $\mathcal{Z}$.

The important consequence is that the Bridgeland-King-Reid theorem works in the situation of diagram (8.1):

Corollary 8.4. *The Fourier-Mukai functor:*

$$\mathbf{\Phi} = \mathbf{R}p_* \circ q^* : \mathbf{D}^b(X^{[n]}) \longrightarrow \mathbf{D}^b_{\mathfrak{S}_n}(X^n)$$

is an equivalence of derived categories.

8.2 Cohomology of representations of tautological bundles

Let X a smooth quasi-projective algebraic surface and let L a line bundle on X. Let $\Xi \subseteq X^{[n]} \times X$ the universal subscheme. It is flat and finite over $X^{[n]}$ of degree n.

Definition 8.5. We call the tautological bundle over $X^{[n]}$ associated to the line bundle L the rank n vector bundle:

$$L^{[n]} := (p_{X^{[n]}})_* p_X^* L$$

where $p_{X^{[n]}}$ and p_X are the projections of Ξ over $X^{[n]}$ and over X respectively.

Tautological bundles on Hilbert schemes are very important for many reasons; they play an important role in the topology of $X^{[n]}$, since their Chern classes are important for understanding the structure of the cohomology ring $H^*(X^{[n]}, \mathbb{Q})$ [28]; moreover in many occasions cohomology computations on moduli spaces of sheaves on surfaces can be reduced to cohomology computations on Hilbert schemes of points [9], [10], [29], [31], [30] where the knowledge of the behaviour of tautological bundles can be necessary.

Notation 8.6. Let $\emptyset \neq I \subseteq \{1, \dots, n\}$ a multi-index; we denote with $p_I : X^n \longrightarrow X^I$ the projection onto the factors in I; let $i_I : X \hookrightarrow X^I$ the diagonal immersion. We denote with L_I the sheaf on X^n defined by: $L_I = p_I^*(i_I)_* L$: it is supported on the diagonal Δ_I. Denote with $\mathcal{C}_L^\bullet$ the complex:

$$0 \longrightarrow \oplus_{i=1}^n L_i \longrightarrow \oplus_{|I|=2} L_I \longrightarrow \dots \longrightarrow L_{\{1,\dots,n\}} \longrightarrow 0\,,$$

where $\oplus_{|I|=p+1} L_I$ is placed in degree p. It is exact in degree $\neq 0$. The group $\mathfrak{S}_n$ acts naturally on each factor $\mathcal{C}_L^p = \oplus_{|I|=p+1} L_I$, making the complex $\mathcal{C}_L^\bullet$ $\mathfrak{S}_n$-equivariant.

As a consequence of corollary (8.4), the cohomology of the Hilbert scheme $H^*(X^{[n]}, F)$ with values in any coherent sheaf F can be obtained as the $\mathfrak{S}_n$-equivariant hypercohomology $\mathbb{H}^*_{\mathfrak{S}_n}(X, \mathbf{\Phi}(F))$ on X^n with values in the image $\mathbf{\Phi}(F)$ of F for the Bridgeland-King-Reid equivalence. We proved

Theorem 8.7 (Scala, 2009, [36])**.** *The image of the tautological bundle $L^{[n]}$ via the BKR equivalence is*

$$\mathbf{\Phi}(L^{[n]}) \simeq \mathcal{C}_L^\bullet$$

in the $\mathfrak{S}_n$-equivariant derived category $\mathbf{D}^b_{\mathfrak{S}_n}(X^n)$. Moreover there is a natural morphism

$$\underbrace{\mathcal{C}_L{}^\bullet \otimes^L \dots \otimes^L \mathcal{C}_L{}^\bullet}_{l\text{-times}} \longrightarrow \mathbf{\Phi}(L^{[n]\otimes l})$$

whose mapping cone is acyclic in degree > 0. This means that $R^q p_* q^*(L^{[n]\otimes l}) = 0$ for all $q > 0$ and in degree zero the morphism: $p_* q^*(L^{[n]})^{\otimes l} \longrightarrow p_* q^*(L^{[n]\otimes l})$ is surjective, the kernel being the torsion subsheaf.

As a consequence, the sheaf $\mathbf{\Phi}(L^{[n]\otimes l}) \simeq p_* q^*(L^{[n]\otimes l})$ can be identified with the $E_\infty^{0,0}$ term of the hyperderived spectral sequence $E_1^{p,q} = \oplus_{i_1+\dots+i_l=p} \mathrm{Tor}_{-q}(\mathcal{C}_L^{i_1}, \dots, \mathcal{C}_L^{i_l})$, associated to l-fold derived tensor product $\mathcal{C}_L{}^\bullet \otimes^L \dots \otimes^L \mathcal{C}_L{}^\bullet$. Working out the term $E_\infty^{0,0}$ of the spectral sequence in all generality is hard, due to evident technical difficulties. Nonetheless, for applications to computations of equivariant cohomology, all we really need is the knowledge of the $\mathfrak{S}_n$-*invariants* $\mathbf{\Phi}((E^{[n]})^{\otimes l})^{\mathfrak{S}_n}$ of the image $\mathbf{\Phi}((E^{[n]})^{\otimes l})$, which can be obtained as the term $\mathcal{E}_\infty^{0,0}$ of the spectral sequence $\mathcal{E}_1^{p,q} = (E_1^{p,q})^{\mathfrak{S}_n}$ of invariants of the original spectral sequence $E_1^{p,q}$. In some lucky cases the new spectral sequence $\mathcal{E}_1^{p,q}$ degenerate at level E_2, and provides the following results.

Theorem 8.8 (Scala, 2009, [36]). *i) Let $a \in X$ and let $\mathcal{J}$ the kernel of the morphism: $S^{n-1}H^*(\mathcal{O}_X) \longrightarrow S^{n-2}H^*(\mathcal{O}_X)$ induced by the morphism $S^{n-2}X \longrightarrow S^{n-1}X$ sending x to $a+x$. The cohomology of the double tensor power $L^{[n]} \otimes L^{[n]}$ of tautological bundles is isomorphic to*

$$H^*(X^{[n]}, L^{[n]} \otimes L^{[n]}) \simeq H^*(L^{\otimes^2}) \otimes \mathcal{J} \oplus H^*(L)^{\otimes 2} \otimes S^{n-2}H^*(\mathcal{O}_X)$$

as $\mathbb{Z}$-graded modules and $\mathfrak{S}_2$-representations.

ii) The cohomology of the general exterior power $\Lambda^k L^{[n]}$ is isomorphic to

$$H^*(X^{[n]}, \Lambda^k L^{[n]}) \simeq \Lambda^k H^*(L) \otimes S^{n-k}H^*(\mathcal{O}_X) \ .$$

8.3 Conclusions

There are many other aspects of McKay correspondence that we could not touch, in connection with valuation theory, string theory, motivic integration, noncommutative geometry, perverse sheaves, Gromov-Witten invariants and quantum cohomology, Donaldson-Thomas invariants, orbifolds, mirror symmetry, for example. See [35].

Research in McKay correspondence is still extremely active: we just mention the crepant resolution conjecture of Chen-Ruan [7], and the derived McKay correspondence conjecture. For the latter it seems that an encouraging direction is the use of moduli spaces of representations of the McKay quiver. See [8].

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